3301. (a) The outputs increase by k: $[a+k, b+k]$.

(b) The outputs are squared: $\left[a^2, b^2\right]$.

(c) The outputs are reciprocated, which reverses their order. And, since $0 \notin [a,b]$, there is no asymptotic behaviour. The range is $[1/b, 1/a]$.

3302. Since all parabolae are stretches and translations of one another, proving the result for $y = x^2$ proves it for all parabolae.

The equation of a generic tangent line at (p, p^2) is $y - p^2 = 2p(x-p)$, which is $y = 2px - p^2$. Another at (q, q^2) has equation $y = 2qx - q^2$. Solving these simultaneously,

$$\begin{aligned} 2px - p^2 &= 2qx - q^2 \\ \Longrightarrow x(2p-2q) &= p^2 - q^2 \\ \Longrightarrow x &= \frac{(p+q)(p-q)}{2(p-q)}. \end{aligned}$$

Assuming that $p \neq q$ (distinct tangent lines), we can divide top and bottom by $(p-q)$, which gives

$$x = \frac{p+q}{2}, \text{ as required.}$$

3303. For f to be decreasing, we require $f'(x) < 0$. The inequality $1 - 2x < 0$ has solution set $(1/2, \infty)$.

Likewise for g, $3x^2 - 12x < 0$. This has boundary equation $x(x-4) = 0$. The solution set is $(0, 4)$.

The intersection of the solution sets is $(1/2, 4)$ or $\{x \in \mathbb{R} : 1/2 < x < 4\}$.

— Nota Bene —

There is no difference in meaning between the set notations $(1/2, 4)$ and $\{x \in \mathbb{R} : 1/2 < x < 4\}$.

Interval set notation is more succinct, but there's nothing wrong with explicit notation. The latter comes into its own when using e.g. rationals $\mathbb{Q}$ or integers $\mathbb{Z}$. The set $\{x \in \mathbb{Q} : 1/2 < x < 4\}$ would be a little clunky in interval notation, requiring an intersection: $(1/2, 4) \cap \mathbb{Q}$.

3304. There are $n!$ orders of the n objects. Consider the first r of these to be those chosen:

$$\underbrace{*****}_{r \text{ chosen}} \mid \underbrace{*******}_{n-r \text{ unchosen}}$$

Each chosen set will be overcounted in the list of $n!$ orders. There are $r!$ arrangements of the chosen set and $(n-r)!$ arrangements of the unchosen set, so each chosen set will be counted $r!(n-r)!$ times. So, we divide by this overcounting factor, giving

$${}^n\mathrm{C}_r = \frac{n!}{r!(n-r)!}.$$

3305. We can integrate by inspection, as the numerator is the derivative of the denominator:

$$\begin{aligned} &\int_{-1}^{1} \frac{3x^2 + 18x + 26}{x^3 + 9x^2 + 26x + 24}\,dx \\ &= \Big[\ln|x^3 + 9x^2 + 26x + 24|\Big]_{-1}^{1} \\ &= \ln 60 - \ln 6 \\ &= \ln 10, \text{ as required.} \end{aligned}$$

— Alternative Method —

To write the integrand in partial fractions, first we factorise the denominator to $(x+2)(x+3)(x+4)$. So, we proposed the following identity:

$$\frac{3x^2 + 18x + 26}{x^3 + 9x^2 + 26x + 24} \equiv \frac{A}{x+2} + \frac{B}{x+3} + \frac{C}{x+4}.$$

Either substituting values or equating coefficients gives $A = B = C = 1$. We can now integrate:

$$\begin{aligned} &\int_{-1}^{1} \frac{3x^2 + 18x + 26}{x^3 + 9x^2 + 26x + 24}\,dx \\ &= \int_{-1}^{1} \frac{1}{x+2} + \frac{1}{x+3} + \frac{1}{x+4}\,dx \\ &= \Big[\ln|x+2| + \ln|x+3| + \ln|x+4|\Big]_{-1}^{1} \\ &= \ln 60 - \ln 6 \\ &= \ln 10, \text{ as required.} \end{aligned}$$

3306. Let the circle has radius 3 and centre O. The stripes have width 2. Solving $x^2 + y^2 = 9$ with $x = 1$ gives $y = \pm\sqrt{8}$. The angle subtended by each shaded segment is $\theta = 2\arctan\sqrt{8}$. The area of each shaded segment is then

$$A_{\text{seg}} = \tfrac{9}{2}(\theta - \sin\theta) = 8.2502...$$

The fraction shaded is

$$\frac{2 \times A_{\text{seg}}}{9\pi} = 0.5835... \approx 58\%, \text{ as required.}$$

3307. (a) This is true. This is a quadratic in $\sin x$ with $\Delta = -3 < 0$.

(b) This is false: $x = \pi$ satisfies the equation.

(c) This is false. This can be seen from the graph of $y = \sin^2 x$ and $y = -\tan x - 1$. The range of the latter for $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$ is $\mathbb{R}$, so the graphs must intersect somewhere in this domain.

3308. Consider the trial in which there is an end result. The possibility space is TTH, with probability $3/8$, and TTT, with probability $1/8$. Out of these, the probability of the latter is $1/4$.

3309. Substituting the latter into the former,

$$\begin{aligned} &(x-2)\left(\tfrac{1}{x}-2\right)=1 \\ \Longrightarrow\ &\tfrac{2}{x}+2x-4=0 \\ \Longrightarrow\ &x^2-2x+1=0 \\ \Longrightarrow\ &(x-1)^2=0. \end{aligned}$$

This equation has a double root at $x=1$, so the hyperbolae are tangent at $(1,1)$.

3310. The implication goes forwards. If f has a factor of $(x-p)^2$, then $y=\mathrm{f}(x)$ has a double root at $x=p$, which is a point of tangency with the x axis, i.e. a stationary point. But e.g. $\mathrm{f}(x)=x^2+1$ has a stationary point at $x=0$, but does not have a factor of x^2.

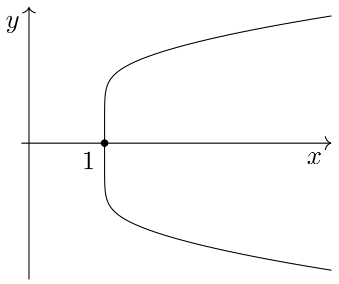
3311. (a) Differentiating with respect to x,

$$3x^2+8y^7\tfrac{dy}{dx}=0.$$

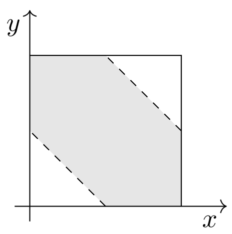
At point $(1,0)$, the derivative $\frac{dy}{dx}$ is undefined, hence the tangent is parallel to the y axis, with equation $x=1$.

(b) Rearranging to $x^3=1+y^8$, we see that, since 8 is even, the RHS is greater than or equal to 1. Hence $x^3\geq 1$, and therefore, since 3 is odd, $x\geq 1$.

(c) The gradient isn't undefined or zero anywhere but $(1,0)$. Also, as $y\to\pm\infty$, $x\to\infty$. The x axis is a line of symmetry. Overall, this gives



3312. The possibility space is the square $[0,1]\times[0,1]$. The successful region is shaded below:



The probability is the area of the shaded region, which is $\frac{3}{4}$.

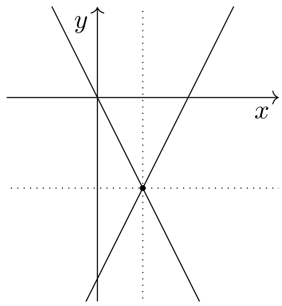
3313. Since $\ln 1=0$, the integral required is

$$A=\int_1^2 \ln x\,dx.$$

We integrate by parts. Let $u=\ln x$ and $\frac{dv}{dx}=1$, so $\frac{du}{dx}=\frac{1}{x}$ and $v=x$. Using the parts formula,

$$\begin{aligned} A&=\Big[x\ln x\Big]_1^2-\int_1^2 \tfrac{1}{x}\cdot x\,dx \\ &=\Big[x\ln x\Big]_1^2-\int_1^2 1\,dx \\ &=\Big[x\ln x-x\Big]_1^2 \\ &=(2\ln 2-2)-(\ln 1-1) \\ &=\ln 4-1. \end{aligned}$$

3314. The points equidistant from L_1 and L_2 lie on the angle bisectors of these lines.



The lines intersect at $(2,-4)$, and their gradients are negatives of each other, so the angle bisectors have equations $x=2$ and $y=-4$. The union of these sets of points can then be expressed with the single equation $(x-2)(y+4)=0$.

3315. (a) Expanding with a compound-angle formula,

$$\begin{aligned} \cos\left(\tfrac{\pi}{2}+A\right)&\equiv\cos\tfrac{\pi}{2}\cos A-\sin\tfrac{\pi}{2}\sin A \\ &\equiv -\sin A. \end{aligned}$$

For small A, we have $\sin A\approx A$, which yields the required result: $\cos\left(\frac{\pi}{2}+A\right)\approx -A$.

(b) Expanding with a compound-angle formula,

$$\begin{aligned} \tan\left(\tfrac{\pi}{4}+A\right)&\equiv\frac{\tan\frac{\pi}{4}+\tan A}{1-\frac{\pi}{4}\tan A} \\ &\equiv\frac{1+\tan A}{1-\tan A}. \end{aligned}$$

For small A, $\tan A\approx A$, which yields

$$\begin{aligned} \tan\left(\tfrac{\pi}{4}+A\right)&\approx\frac{1+A}{1-A} \\ &\equiv\frac{1-A+2A}{1-A} \\ &\equiv 1+\frac{2A}{1-A},\ \text{as required.} \end{aligned}$$

3316. (a) Setting $y=0$, we take out a factor of $x^{\frac{1}{3}}$ to give $x^{\frac{1}{3}}(x^{\frac{1}{3}}+1)=0$. So, $x=0,-1$.

(b) Differentiating,

$$\begin{aligned} & y = x^{\frac{2}{3}} + x^{\frac{1}{3}} \\ \Longrightarrow\ & \tfrac{dy}{dx} = \tfrac{2}{3}x^{-\frac{1}{3}} + \tfrac{1}{3}x^{-\frac{2}{3}} \\ \Longrightarrow\ & \tfrac{d^2y}{dx^2} = -\tfrac{2}{9}x^{-\frac{4}{3}} - \tfrac{2}{9}x^{-\frac{5}{3}}. \end{aligned}$$

Setting the first derivative to zero for SPs,

$$\tfrac{2}{3}x^{-\frac{1}{3}} + \tfrac{1}{3}x^{-\frac{2}{3}} = 0.$$

Multiplying by $x^{\frac{2}{3}}$ gives $x^{\frac{1}{3}} = -\frac{1}{2}$, so $x = -\frac{1}{8}$. Substituting into the second derivative gives $64/9 > 0$. So, there is one SP, a local minimum at $(-1/8, -1/4)$.

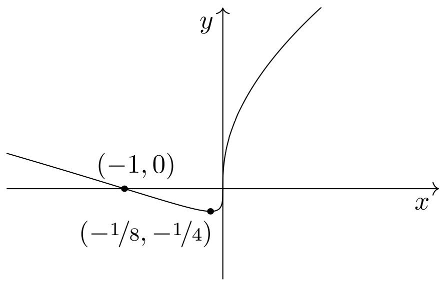
(c) At $(0, 0)$, the curve is defined, but calculating the first derivative would involve division by zero. This signifies a tangent parallel to the y axis. Since this is at O, the y axis itself is tangent to the curve.

(d) The curve is concave whenever the second derivative is negative:

$$\begin{aligned} & -\tfrac{2}{9}x^{-\frac{4}{3}} - \tfrac{2}{9}x^{-\frac{5}{3}} < 0 \\ \Longrightarrow\ & x^{-\frac{4}{3}} + x^{-\frac{5}{3}} > 0. \end{aligned}$$

Solving the boundary equation gives $x = -1$. The second derivative is also undefined (and changes sign) at $x = 0$. The second derivative is negative outside these values, so it is concave on $(-\infty, -1) \cup (0, \infty)$ as required.

(e) Putting all of the above information together with the fact that, as $x \to \pm\infty$, $y \to \infty$, the graph is



——— Nota Bene ———

The curve is only very slightly concave for $x \in (-\infty, -1)$, which is why the curve looks like a straight line for negative x.

3317. There are ${}^{10}\text{C}_2 = 45$ ways of selecting two distinct integers. Of these, there are 9 sets (the lower value of the two being $1, 2, ..., 9$) whose difference is 1. So, the probability is $1/5$.

3318. (a) Since the string forms a closed loop, it can be tightened to any extent without affecting the dynamics of the masses. From the information given, we have no way of knowing the extent to which the string has been tightened. The tensions could, as a result, be arbitrarily high.

(b) Setting up the equation of motion along the string, all of the tensions cancel, and the only relevant forces are the weights of the 1 and 2 kg masses: $2g - g = (2+3+1)a \implies a = \frac{1}{6}g$.

3319. By the quotient rule, the first derivative is

$$\frac{dy}{dx} = \frac{3k(1-x^2)}{(x^2+1)^3}.$$

So, the curve is stationary where $3k(1-x^2) = 0$. We know $k \neq 0$, as otherwise $y = 0$ everywhere. Hence, $1 - x^2 = 0$, so $x \pm 1$. Substituting these into the original equation gives $\pm\frac{k}{16} = \pm 1$. The $\pm$ signs could combine either way, so $k = \pm 16$.

3320. Squaring and rearranging,

$$\begin{aligned} & \sqrt{2x+1} + \sqrt{2x-2} = 3 \\ \Longrightarrow\ & 2x + 1 + 2\sqrt{(2x+1)(2x-2)} + 2x - 2 = 9 \\ \Longrightarrow\ & \sqrt{(2x+1)(2x-2)} = 5 - 2x. \end{aligned}$$

Squaring again,

$$\begin{aligned} & (2x+1)(2x-2) = (5-2x)^2 \\ \Longrightarrow\ & 4x^2 - 2x - 2 = 25 - 20x + 4x^2 \\ \Longrightarrow\ & 18x = 27 \\ \Longrightarrow\ & x = \tfrac{3}{2}. \end{aligned}$$

This does indeed satisfy the original equation, so the solution is $x = \frac{3}{2}$.

——— Nota Bene ———

Whenever squaring both sides of an equation, one must then check any roots found. This is because e.g. $-1 \neq 1$, but $(-1)^2 = 1^2$.

3321. The shortest distance between two curves lies along the normal to both. In the case of circles, this is the line of centres. The centres in question are $(0, -1)$ and $(1/2, 0)$. So, the equation of the shortest path is $y = 2x - 1$.

3322. Consider the boundary equation $\sin\left(x^2\right) = 1/2$. This has roots at

$$\begin{aligned} x &= \sqrt{\pi/6} \approx 0.72, \\ x &= \sqrt{5\pi/6} \approx 1.62. \end{aligned}$$

The inequality is satisfied for all points between these roots. And there are always infinitely many rational numbers in any such interval: for example, $x = 1, 1.1, 1.11, 1.111, ...$ This proves the result.

3323. For the curve's x intercepts,

$$\begin{aligned} & \ln\left(3x^2 + x - 1\right) = 0 \\ \Longrightarrow\ & 3x^2 + x - 1 = 1 \\ \Longrightarrow\ & x = -1, \tfrac{2}{3}. \end{aligned}$$

To find the x axis intercept of the tangent, we can use the Newton-Raphson iteration, with $x_0 = 2.3$. This is

$$x_1 = 2.3 - \frac{f(2.3)}{f'(2.3)} = -0.9985.$$

This is close to $x = -1$, as required.

3324. To be invertible on the domain $\mathbb{R}$, a cubic must have no turning points. This means it can have a maximum of one stationary point. Differentiating, $f'(x) = 3x^2 + a$. For this to have a maximum of one root, we require $x \geq 0$. There is no restriction on b. So, $a \in [0, \infty)$ and $b \in \mathbb{R}$.

3325. We can rewrite the integrand as

$$\begin{aligned} &\frac{x^2-4x+1}{x^2-4x+4} \\ \equiv\ &\frac{x^2-4x+4-3}{x^2-4x+4} \\ \equiv\ &1 - \frac{3}{x^2-4x+4} \\ \equiv\ &1 - \frac{3}{(x-2)^2}. \end{aligned}$$

This gives

$$\int \frac{x^2-4x+1}{x^2-4x+4}\,dx = x + \frac{3}{x-2} + c.$$

3326. This is true of all polynomials up to degree 3. But $y = x^4 + x$, which is neither even nor odd, is a counterexample to the general statement.

Nota Bene

The set of counterexamples to this claim is well defined. It is the set of all polynomials of degree at least four, which contain at least one term of even degree and at least one term of odd degree.

3327. (a) The legs form a pyramid with an equilateral triangle E, of side length $10\sqrt{3}$ cm, as the base. The distance from the centre of this triangle to its vertices is 10 cm. So, each leg is the hypotenuse of a vertical right-angled triangle with hypotenuse 100 cm and opposite side 10 cm. This gives $\theta = \arcsin\frac{1}{10}$.

(b) The resultant of the thrusts is $99g$ N. Their horizontal components must cancel due to the symmetry of E. Vertically, $T\cos\theta = 33g$. And

$$\cos\theta = \sqrt{1-\sin^2\theta} = \tfrac{\sqrt{99}}{10}.$$

This gives $T = 33g \times \frac{10}{\sqrt{99}} = g\sqrt{1100}$.

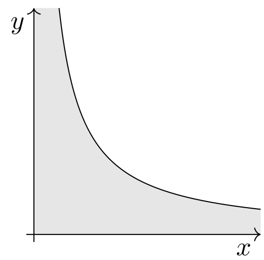
3328. For large a, the triangle approaches equilateral, so cannot have an obtuse angle. The boundary case is one in which the largest angle is a right-angle. In this case, Pythagoras gives

$$\begin{aligned} &(a+1)^2 = a^2 + (a-1)^2 \\ \Longrightarrow\ &a = 0, 4. \end{aligned}$$

We ignore $a = 0$, which gives a negative length. Hence, for T to have an obtuse angle, we require $a < 4$. □

3329. The number of ways of choosing 2 squares from 9 is ${}^9C_2 = 36$. From this possibility space, there are 6 pairs of vertically adjacent squares and 6 pairs of horizontally adjacent squares. So, the probability is $\frac{12}{36} = \frac{1}{3}$.

3330. (a) With $p = q = 1$, the region is as follows:



Other values of p and q give a qualitatively similar graph in the positive quadrant.

Rearranging $x^p y^q = 1$, we get $y = x^{-\frac{p}{q}}$. To see whether region R has finite area, we need to integrate this from 0 to ∞. Neither of these values can be used directly, so each requires calculation of a limit. One way of doing this is to set up a and $\frac{1}{a}$. That way, letting $a \to 0$ sends the lower boundary of the integral to 0 and the upper boundary to ∞, as required.

(b) Evaluating the integral,

$$\begin{aligned} \int_a^{\frac{1}{a}} x^{-\frac{p}{q}}\,dx &\equiv \left[\tfrac{q}{p+q}x^{-\frac{p+q}{q}}\right]_a^{\frac{1}{a}} \\ &\equiv \tfrac{q}{p+q}\left(\tfrac{1}{a}^{-\frac{p+q}{q}} - a^{-\frac{p+q}{q}}\right) \\ &\equiv \tfrac{q}{p+q}\left(a^{\frac{p+q}{q}} - a^{-\frac{p+q}{q}}\right). \end{aligned}$$

Taking out a factor of $\frac{q}{p+q}$ from the limit,

$$A = \tfrac{q}{p+q}\lim_{a\to 0}\left(a^{\frac{p+q}{q}} - a^{-\frac{p+q}{q}}\right).$$

(c) The terms in the bracket above are reciprocals. Hence, as $a \to 0$, exactly one of them tends to zero, and the other tends to infinity. The sum of two such terms must tend to infinity. QED.

3331. In harmonic form $R\sin(2\theta + \alpha)$, the denominator would have $R = \sqrt{8^2 + 15^2} = 17$. So, the range of the denominator is $[-17, 17]$. This contains zero. When reciprocated and scaled by 34, the range of $g(x)$ is $(-\infty, -2] \cup [2, \infty)$.

3332. We can take f to be a positive polynomial wlog.

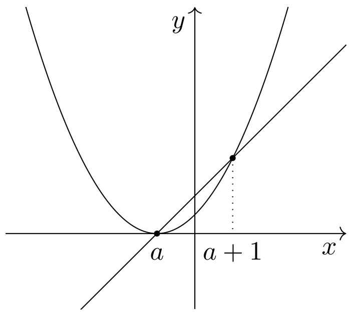
Assume, for a contradiction, that f is a polynomial function of even degree with $f'(x) > 0$ for all x.

Then $f'(x) = 0$ has no roots, so the function is not stationary for any $x \in \mathbb{R}$. Hence, it has no turning points. But a positive polynomial function of even degree must have at least one turning point, since, as $x \to \pm\infty$, $y \to \infty$. This is a contradiction. Hence, no polynomial function of even degree can be increasing everywhere. $\square$

3333. The boundary equation is

$$\begin{aligned} &(x-a) = (x-a)^2 \\ \Longrightarrow\ &(x-a)^2 - (x-a) = 0 \\ \Longrightarrow\ &(x-a)(x-a-1) = 0 \\ \Longrightarrow\ &x = a, a+1. \end{aligned}$$

Sketching the LHS and RHS,



We need the x values for which the straight line is at or above the parabola. This is $x \in [a, a+1]$.

3334. The GP is a, ar, ar^2, ar^3. So, we need to show that $r^3 + 1 > r^2 + r$. Solving the boundary equation,

$$\begin{aligned} &r^3 + 1 = r^2 + r \\ \Longrightarrow\ &r = \pm 1. \end{aligned}$$

We are told that the GP is increasing. For $r > 1$, the boundary equation has no roots, and so the quantities $r^3 + 1$ and $r^2 + r$ are never equal.

Furthermore, the cubic is always larger than the quadratic, as seen in, for example, $r = 2$. Hence, the inequality $r^3 + 1 > r^2 + r$ holds, which gives $a + d > b + c$. $\square$

Nota Bene

An example such as $r = 2$ cannot, in general, prove a broader result. In this case, however, the LHS and RHS are polynomials which are never equal for $r > 1$. So, if the LHS is greater than the RHS for *any* value $r > 1$, then the LHS is greater than the RHS for *all* values $r > 1$.

3335. Without the condition $XYZ = 6$, the probability is $1/6$. With it, the possibility space of $6^3 = 216$ outcomes is restricted to the $3! = 6$ permutations of $\{1, 2, 3\}$ and the 3 permutations of $\{1, 1, 6\}$. Of these, four have $X = 1$. So, the probability has increased from $1/6$ to $4/9$.

3336. The input of each logarithm is

$$1 + \tfrac{1}{i} \equiv \frac{i+1}{i}.$$

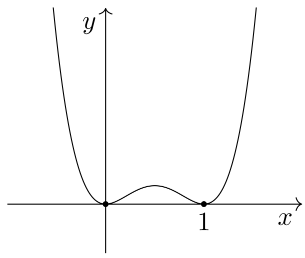
A log rule tells us that a sum of logarithms is the logarithm of a product. Expanding this, all of the factors cancel but $n + 1$:

$$\begin{aligned} &\sum_{i=1}^{n} \ln\left(1 + \frac{1}{i}\right) \\ =\ &\ln\left(\tfrac{2}{1} \cdot \tfrac{3}{2} \cdot \ldots \cdot \tfrac{n+1}{n}\right) \\ =\ &\ln(n+1), \text{ as required.} \end{aligned}$$

3337. Substituting for q,

$$\begin{aligned} p &= (x-1)^2 + (x-1) \\ &\equiv x(x-1). \end{aligned}$$

Substituting for p gives $y = x^2(x-1)^2$. This is a positive quartic, with double roots at $x = 0, 1$:



3338. Differentiating implicitly with respect to x,

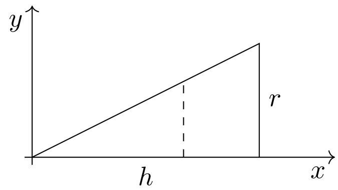
$$1 = 2y\frac{dy}{dx}.$$

So, at $(16, 4)$, the curve has gradient $\frac{1}{8}$. Hence, the normal has equation

$$y - 4 = -8(x - 16).$$

Solving simultaneously with $x = y^2$ gives roots $y = 4$ (as in the question) and $y = -4.125$.

3339. (a) In (x, y) half-section, the cone is



The line shown has equation $y = \frac{r}{h}x$, so the quantity in question is y, the length of the dashed line above. This becomes the radius of the circular cross-section when the line is rotated around the x axis.

(b) The integrand is an expression for the area of the circular cross-section of the cone.

(c) Evaluating the integral (the continuous sum of all such circular cross-sections),

$$\begin{aligned} V_{\text{cone}} &= \int_0^h \pi\left(\tfrac{r}{h}x\right)^2\,dx \\ &= \frac{\pi r^2}{h^2}\int_0^h x^2\,dx \\ &= \frac{\pi r^2}{h^2}\times\tfrac{1}{3}h^3 \\ &= \tfrac{1}{3}\pi r^2 h, \text{ as required.} \end{aligned}$$

3340. ② is larger. The greatest and least values deviate from the mean by less than 1, which means that the deviation from the mean is less that 1 for every datum. Squaring such deviations reduces them, so $(x_i-\bar{x})^2 < |x_i-\bar{x}|$ for every x_i. Therefore, since all terms are positive in both sums, the second sum is larger than the first.

3341. The x intercepts of the first parabola are $\{-4, 3\}$ and of the second parabola are $\{0, -b\}$. For the first set to be transformed to the second set by a reflection in $x = a$, the value must be halfway between each x intercept and its image.

- If -4 maps to 0, then $a = -2$ and $b = -7$.
- If 3 maps to 0, then $a = 1.5$ and $b = 7$.

3342. A scale factor from the chain rule has been missed. The numerator is $2 \times 2x$, which is $2\,\mathrm{f}'(x)$, where $\mathrm{f}(x) = x^2 + 1$. Hence, quoting the standard result,

$$\int \frac{4x}{x^2+1}\,dx = 2\ln|x^2+1| + c.$$

Nota Bene

To get a deeper understanding of this, as ever with inspections, differentiate the answer.

3343. Let $y = \arcsin x$, so that $x = \sin y$. This gives

$$\begin{aligned} \tan(\arcsin x) &= \tan y \\ &\equiv \frac{\sin y}{\cos y} \\ &= \frac{\sin y}{\sqrt{1-\sin^2 y}} \\ &= \frac{x}{\sqrt{1-x^2}}. \end{aligned}$$

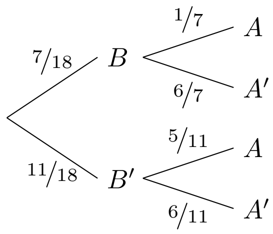
Nota Bene

We are justified in taking the positive square root in the above by the fact that the arcsin function has domain $[-\pi/2, \pi/2]$. Over this domain, cosine is non-negative.

3344. The calculations for the top branches are

$$\begin{aligned} \mathbb{P}(B) &= \tfrac{1}{3}\times\tfrac{1}{6}+\tfrac{2}{3}\times\tfrac{1}{2} = \tfrac{7}{18} \\ \mathbb{P}(A \mid B) &= \frac{\frac{1}{3}\times\frac{1}{6}}{\frac{1}{3}\times\frac{1}{6}+\frac{2}{3}\times\frac{1}{2}} = \tfrac{1}{7}. \end{aligned}$$

Calculating the other branches similarly, the tree diagram is



3345. (a) Differentiating,

$$\begin{aligned} & \mathrm{g}(x) = x^{2k} + x^{2k+1} \\ \Longrightarrow\ & \mathrm{g}'(x) = 2kx^{2k-1} + (2k+1)x^{2k} \\ \Longrightarrow\ & \mathrm{g}''(x) = 2k(2k-1)x^{2k-2} \\ & \qquad + (2k+1)2kx^{2k-1}. \end{aligned}$$

Taking out a common factor of $2kx^{2k-2}$,

$$\mathrm{g}''(x) = 2kx^{2k-2}\big((2k-1)+(2k+1)x\big).$$

(b) At $x = 0$, the linear factor is non-zero (thus doesn't change sign) because $2k - 1 \geq 3$ and $2k + 1 \geq 5$. And the factor x^{2k-2} is raised to an even power, so it doesn't change sign. Hence, while $\mathrm{g}''(0) = 0$, $x = 0$ is not a point of inflection of $y = \mathrm{g}(x)$.

3346. Under rotation by 180° about the point $(1, 0)$, the interval $[0, 1]$ maps to the interval $[1, 2]$, and maps y to $-y$. An integral with respect to x calculates the signed area between the curve and the x axis. Hence, the results are

(a) $\displaystyle\int_0^2 \mathrm{f}(x)\,dx = 0,$

(b) $\displaystyle\int_2^1 \mathrm{f}(x)\,dx = -\int_1^2 \mathrm{f}(x)\,dx = k.$

3347. Since the dust particle is accelerating vertically, along the line of action of the weight, the resultant of the electromagnetic forces must also be vertical. Using Pythagoras, the resultant force is

$$\begin{aligned} &\sqrt{(6.6\times10^{-5})^2+(1.4\times10^{-4})^2} - 13\times10^{-6}g \\ &= 2.73773\times10^{-5}\text{ N}. \end{aligned}$$

NII gives $a = \dfrac{2.73773\times10^{-5}}{13\times10^{-6}} = 2.11\text{ ms}^{-2}.$

Alternative Method

Call the angle between the right-hand force and the horizontal θ. Horizontal equilibrium gives

$$\begin{aligned} & 1.4\times 10^{-4}\cos\theta = 6.6\times 10^{-5}\sin\theta \\ \Longrightarrow\ & \tan\theta = \tfrac{70}{33} \\ \therefore\ & \theta = \arctan\tfrac{70}{33}. \end{aligned}$$

Resolving vertically,

$$\begin{aligned} & 1.4\times 10^{-4}\sin\theta + 6.6\times 10^{-5}\cos\theta \\ & \qquad - 13\times 10^{-6}g = 13\times 10^{-6}a \\ \Longrightarrow\ & a = 2.11 \text{ ms}^{-2} \text{ (3sf)}. \end{aligned}$$

3348. Let $x = 1$. Substituting this in,

$$\begin{aligned} & y^3 - 3y - 2 = 0 \\ \Longrightarrow\ & y = -1, 2. \end{aligned}$$

Since there are two y values corresponding to one x value, the relationship is many-to-one. Hence, it cannot be written in the form $y = \mathrm{f}(x)$, where f is a function.

3349. The possibility space is an $m\times n$ rectangle. The set of outcomes for which the score on the m-sided die is larger is triangular, with sides of length $m-1$. The number of outcomes it contains is

$$\begin{aligned} & 1 + 2 + ... + m - 1 \\ \equiv\ & \tfrac{1}{2}m(m-1). \end{aligned}$$

So, the probability p that the score on the m-sided die is larger is given by

$$p = \frac{\frac{1}{2}m(m-1)}{mn} = \frac{m-1}{2n}.$$

3350. If integers differ by an odd number, then one of them must be even. If this is to be prime, then it must be 2. So, consider $2+3=5$, $2+5=7$ and $2+5=9$. The first two are counterexamples to (a) and (b). But, since 9 is non-prime, statement (c) is true.

(a) False,

(b) False,

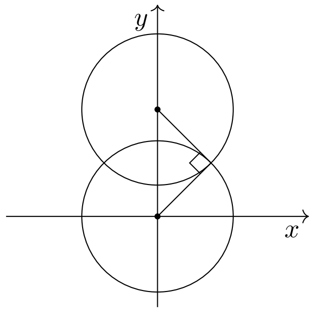
(c) True.

3351. (a) Each side has endpoints which are a vertex and a midpoint. Hence, they are all the same length, and the shape is a rhombus.

(b) The diagonals have lengths $|AB| = \sqrt{3}$ and $|MN| = \sqrt{2}$. So,

$$A_{\text{rhombus}} = \tfrac{1}{2}\times\sqrt{3}\times\sqrt{2} = \tfrac{\sqrt{6}}{2}.$$

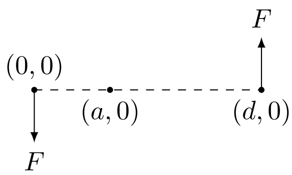
3352. The scenario is



The centre of the upper circle is at $\left(0, \sqrt{2}\right)$.

The diagram can also be reflected in the x axis, so $k = \pm\sqrt{2}$.

3353. We can place the forces and the pivot point on an x axis, without loss of generality. Consider a force $-F\mathbf{j}$ at $(0,0)$ and a force $F\mathbf{j}$ at $(d,0)$:

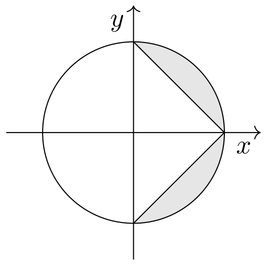


The resultant anticlockwise moment around $(a,0)$ is $Fa + F(d-a) = Fd$. Since this is independent of a, the turning effect is the same around any pivot point. □

3354. Using a double-angle formula, the integrand can be simplified as $4x\sin 2x$. We can now integrate by parts, with $u = 2x$ and $\frac{dv}{dx} = 2\sin 2x$. These give $\frac{du}{dx} = 2$ and $v = -\cos 2x$. Substituting into the parts formula,

$$\begin{aligned} & \int 8x\cos x\sin x\,dx \\ =\ & -2x\cos 2x + \int 2\cos 2x\,dx \\ =\ & -2x\cos 2x + \sin 2x + c. \end{aligned}$$

3355. Region R consists of two equal segments:



Each of these has area $\frac{\pi}{4} - \frac{1}{2}$, so the total area is $\frac{\pi}{2} - 1$, as required.

3356. The intersections are at

$$\begin{aligned} x &= (2x^2-1)^2 \\ \Longrightarrow 4x^4 &- 4x^2 - x + 1 = 0. \end{aligned}$$

A polynomial solver gives $x = 1$ or $0.419643....$ So, the second y coordinate is $0.419643^2 = 0.1761$ (4sf). The area enclosed is

$$\begin{aligned} &\int_{0.1761}^{1} y^{\frac{1}{2}} - (2y-1)^2 \, dy \\ &= \left[\tfrac{2}{3}y^{\frac{3}{2}} - \tfrac{1}{6}(2y-1)^3\right]_{0.1761}^{1} \\ &= 0.5 - 0.09457 \\ &= 0.405 \text{ (3sf)}. \end{aligned}$$

3357. (a) Reversing the order of the limits negates the value of a definite integral. Integrating term by term,

$$\begin{aligned} &\int_b^a \mathrm{f}(x) - 1 \, dx \\ \equiv &\int_a^b 1 - \mathrm{f}(x) \, dx \\ &= b - a - I. \end{aligned}$$

(b) We integrate by substitution, letting $u = 3x$. The new limits are $u = a$ and $u = b$. The ratio of infinitesimals is $\frac{1}{3}du = dx$. This gives

$$\begin{aligned} &\int_{\frac{b}{3}}^{\frac{a}{3}} \mathrm{f}(3x) \, dx \\ &= \tfrac{1}{3}\int_a^b \mathrm{f}(u) \, du \\ &= \tfrac{1}{3}I. \end{aligned}$$

3358. Using the generalised binomial expansion,

$$(1-x)^{\frac{1}{2}} = 1 - \tfrac{1}{2}x - \tfrac{1}{8}x^2 + ...$$

Let $x = 0.02$. This is (very) small, so we can ignore terms in x^2 and above. This gives

$$\begin{aligned} &\sqrt{1-0.02} \approx 1 - \tfrac{1}{2} \times 0.02 \\ \Longrightarrow &\sqrt{\tfrac{98}{100}} \approx \tfrac{99}{100} \\ \Longrightarrow &\tfrac{7}{10}\sqrt{2} \approx \tfrac{99}{100} \\ \Longrightarrow &\sqrt{2} \approx \tfrac{99}{70}, \text{ as required.} \end{aligned}$$

3359. An iteration $x_{n+1} = \mathrm{g}(x_n)$ has a fixed point iff $x = \mathrm{g}(x)$. Setting this equation up,

$$\begin{aligned} &x = \sum_{r=1}^{2k+1} x^r \\ \Longleftrightarrow\ &x = x + x^2 + ... + x^{2k+1} \\ \Longleftrightarrow\ &x^{2k+1} + x^{2k} + ... + x^3 + x^2 = 0 \\ \Longleftrightarrow\ &x^2(x^{2k-1} + x^{2k-2} + ... + x + 1) = 0. \end{aligned}$$

This is satisfied if $x = 0$ or if

$$x^{2k-1} + x^{2k-2} + ... + x + 1 = 0.$$

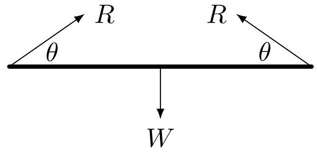
Since $k \in \mathbb{N}$, this is a polynomial equation of odd degree, so it must have at least one real root. And that root is not $x = 0$. Hence, I has at least two fixed points, as required.

3360. Multiplying up by $\sin 2x$,

$$\begin{aligned} &\operatorname{cosec} 2x + \sec 2x = 0 \\ \Longrightarrow\ &1 + \tan 2x = 0 \\ \Longrightarrow\ &\tan 2x = -1 \\ \Longrightarrow\ &2x = \tfrac{3\pi}{4}, \tfrac{7\pi}{4}, \tfrac{11\pi}{4}, \tfrac{15\pi}{4} \\ \Longrightarrow\ &x = \tfrac{3\pi}{8}, \tfrac{7\pi}{8}, \tfrac{11\pi}{8}, \tfrac{15\pi}{8}. \end{aligned}$$

3361. (a) $\theta = \arctan \frac{1}{2}$ is the angle in a $(1, 2, \sqrt{5})$ right-angled triangle. So, $\sin\theta = 1/\sqrt{5}$, as required.

(b) The bowl is smooth, so the force exerted at each point of contact acts along the normal to the curve. The points of contact are at $(\pm 1, 1)$, at which point the gradient of the curve is ± 2, so the gradient of the normal is $\mp\frac{1}{2}$. This gives the angle of inclination of each contact force as $\theta = \arctan \frac{1}{2}$.



Resolving vertically,

$$\begin{aligned} &2R\sin(\arctan \tfrac{1}{2}) = W \\ \Longrightarrow\ &2R \cdot \tfrac{1}{\sqrt{5}} = W \\ \Longrightarrow\ &R = \tfrac{\sqrt{5}}{2}W. \end{aligned}$$

By NIII, this is the same as the contact force exerted by each end on the bowl.

3362. Call the two cubics $y = \mathrm{f}(x)$ and $y = \mathrm{g}(x)$. Since the cubics are tangent, $\mathrm{f}(x) - \mathrm{g}(x) = 0$ must have a repeated root. And, since they cross at the point of tangency, this cannot be a double root. Hence, it must be a triple root, since $\mathrm{f}(x) - \mathrm{g}(x)$ is cubic. This exhausts the roots of $\mathrm{f}(x) - \mathrm{g}(x) = 0$. Hence, the curves cannot cross again elsewhere. □

3363. The term quadratic in z must be

$$(x^2 + x + 1)^2 \equiv x^4 + 2x^3 + 3x^2 + 2x + 1.$$

This matches the terms in x^4 and x^3. We then require $-2(x^2 + x + 1)$ for the term in x^2. This matches all remaining terms. So, we have

$$x^4 + 2x^3 + x^2 - 1 = z^2 - 2z.$$

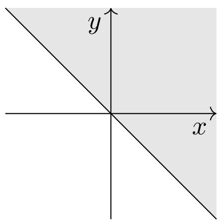
3364. Using the cosine rule,

$$c^2 = 2r^2 - 2r^2\cos\theta.$$

We differentiate implicitly with respect to t, noting that the radius r is constant:

$$\begin{aligned}&2c\frac{dc}{dt} = 2r^2\sin\theta\frac{d\theta}{dt}\\ \Longrightarrow\ &\frac{dc}{dt} = \frac{r^2}{c}\sin\theta\frac{d\theta}{dt}, \text{ as required.}\end{aligned}$$

3365. The factor $|x-y|$ is always greater than or equal to zero, so the inequality is satisfied iff $x+y \geq 0$. This is the region above the straight line $x+y=0$:



3366. Using the generalised binomial expansion,

$$\begin{aligned}&\frac{x}{1-x^2}\\ &\equiv x\left(1-x^2\right)^{-1}\\ &= x\left(1 + (-1)\left(-x^2\right) + \tfrac{(-1)(-2)}{2!}\left(-x^2\right)^2 + ...\right)\\ &\equiv x + x^3 + x^5 + ...\end{aligned}$$

The binomial expansion is valid for $|x| < 1$, and the quintic approximation holds for x close to zero.

— Nota Bene —

To clarify the last sentence: further to the *validity* of the expansion, which is for $|x| < 1$, there is also the question of the *quality* of the approximation. These aren't quite the same thing.

① The *validity* of the expansion says whether, given enough terms, it is possible to find an arbitrarily good approximation. That is, it determines whether or not the infinite series converges.

② The *quality* of an approximation differs, even where the expansion is valid. For x close to 1, the higher powers of x cannot be ignored, as we have done. The quintic approximation is of higher quality closer to $x = 0$.

3367. Call the angle of projection θ. Vertically, looking for the time of flight,

$$\begin{aligned}0 &= u\sin\theta t - \tfrac{g}{2}t^2\\ \therefore\quad t &= \frac{2u\sin\theta}{g}.\end{aligned}$$

So, the horizontal range is given by

$$d = \frac{2u^2\sin\theta\cos\theta}{g}.$$

The boundary case, therefore, is

$$\begin{aligned}&\frac{2u^2\sin\theta\cos\theta}{g} = \frac{u^2}{2g}\\ \Longrightarrow\ &2\sin\theta\cos\theta = \tfrac{1}{2}\\ \Longrightarrow\ &\sin 2\theta = \tfrac{1}{2}\\ \Longrightarrow\ &2\theta = 30°, 150°\\ \Longrightarrow\ &\theta = 15°, 75°.\end{aligned}$$

The success space is $[15°, 75°]$, which is $2/3$ of the possibility space. So, the probability is $2/3$.

3368. Integrating by inspection,

$$\int e^{\sin t}\cos t\,dt = e^{\sin t} + c.$$

— Nota Bene —

An inspection is best understood backwards, by differentiating the result.

3369. At $x = p$, the gradient is $m = 2p$, so the normal has gradient $-1/2p$. This gives the equation of the normal at $\left(p, p^2\right)$ as

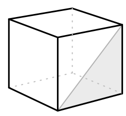
$$\begin{aligned}&y - p^2 = -\tfrac{1}{2p}(x-p)\\ \Longrightarrow\ &y = -\tfrac{1}{2p}x + 1 + p^2.\end{aligned}$$

The y intercept is $c = 1 + p^2$, which, since $p \neq 0$, is greater than 1, as required.

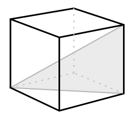
3370. By the SSS condition, if their sets of side lengths are the same, then two triangles are congruent. The possible side lengths between vertices of a unit cube are 1 for edges, $\sqrt{2}$ for face diagonals and $\sqrt{3}$ for space diagonals.

- If an edge (of length 1) is involved, then the triangle must be right-angled. So, its sides obey Pythagoras. This gives two triangles:

$(1, 1, \sqrt{2})$

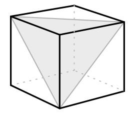


$(1, \sqrt{2}, \sqrt{3})$



- If no edge is included, then there must be at least one face diagonal. From an endpoint of this face diagonal, it is impossible to draw a space diagonal without including an edge. So, the only triangle is all face diagonals:

$(\sqrt{2}, \sqrt{2}, \sqrt{2})$



Overall, there are three triangles, as required.

3371. (a) $p = {}^3\text{C}_1 \times \frac{3}{20} \times \frac{2}{19} \times \frac{17}{18} = \frac{17}{380}$.

(b) Expected payout is $\frac{17}{380} \times £5 = £0.224$ (3sf).

— Nota Bene —

There is no reason to round this value to 0.22, reflecting 22 pence. It is an average (a mean), so fractions of a penny are correct.

(c) $p = \frac{3}{20} \times \frac{2}{19} \times \frac{1}{18} = \frac{1}{1140}$.

(d) The expected payout for the first prize is $\frac{n}{1140}$. So, we want $\frac{n}{1140} = 0.224$, which gives $n = 255$. So, a top prize of £250 might be appropriate.

3372. Assume, for a contradiction, that $n+1$ distinct points on a polynomial graph $y = \text{f}(x)$ of degree n are collinear.

Let $y = mx + c$ be the equation of the relevant line. Consider the equation $\text{f}(x) = mx + c$. This is a polynomial with degree n. But we are told that it has $n+1$ distinct roots. This is a contradiction. Hence, no $n+1$ distinct points on a polynomial graph $y = \text{f}(x)$ of degree $n \geq 2$ are collinear. □

3373. The peg is modelled as smooth, which means that there are no frictional forces which can generate an imbalance in tension: the tension must be the same either side of the peg. Also, the string is light, so the tension must be same throughout the string.

Consider a small piece of string at one end. Since this is light, the external force applied to its end must equal the tension applied to the other end. Hence, the forces applied to each end of the string must each be equal to the tension, and hence equal to each other.

3374. (a) This is true. If x is a fixed point of both then $\text{fg}(x) = \text{f}(x) = x$.

(b) This is false. A counterexample is $\text{f}(x) = 1 - x$ and $\text{g}(x) = -1 + x$. Both have a root at $x = 1$, but $\text{fg}(1) = \text{f}(0) = 1$.

(c) This is false. For a counterexample, consider any functions F and G whose derivatives are

$$\begin{aligned} \text{F}'(x) &= \text{f}(x) = 1 - x, \\ \text{G}'(x) &= \text{g}(x) = -1 + x. \end{aligned}$$

The logic of part (b) then applies.

3375. We can express the ellipse as

$$\left(\sqrt{2}x\right)^2 + \left(\sqrt{5}y\right)^2 = 10.$$

This is the circle $x^2 + y^2 = 10$, with x replaced by $\sqrt{2}x$ and y replaced by $\sqrt{5}y$. The circle has been stretched by scale factor $1/\sqrt{2}$ in the x direction and by scale factor $1/\sqrt{5}$ in the y direction. The original area was 10π. Hence, the new area is

$$10\pi \times \tfrac{1}{\sqrt{2}} \times \tfrac{1}{\sqrt{5}} = \sqrt{10}\pi.$$

3376. Using a Pythagorean identity, the LHS is

$$\frac{2\tan^2 x}{\tan^2 x + 1} \equiv \frac{2\tan^2 x}{\sec^2 x} \equiv 2\sin^2 x$$

Using a double-angle identity, the RHS is

$$1 - \cos 2x \equiv 1 - (1 - 2\sin^2 x) \equiv 2\sin^2 x.$$

So, LHS ≡ RHS, proving the result.

3377. (a) Vertically, $v = 10 - 10t$, so $t = 1$. At this point, the vertical position is $z = 10t - 5t^2 = 5$, so the coordinates are $(20, 20, 5)$.

(b) By symmetry over horizontal ground, the speed with which the particle lands is the same as that with which it was projected. This is $\sqrt{20^2 + 20^2 + 10^2} = 30 \text{ ms}^{-1}$.

(c) By symmetry, the angle below horizontal at landing is the same as the angle of projection. Horizontal speed is $\sqrt{20^2 + 20^2} = 20\sqrt{2}$, so

$$\begin{aligned} \tan\theta &= \tfrac{10}{20\sqrt{2}} \\ \therefore \quad \theta &= 19.47° \text{ (2dp)}. \end{aligned}$$

3378. (a) Differentiating implicitly with respect to y,

$$\begin{aligned} 3y^2 &= \tfrac{dx}{dy} - 2x\tfrac{dx}{dy} \\ \Longrightarrow 3y^2 &= (1 - 2x)\tfrac{dx}{dy}. \end{aligned}$$

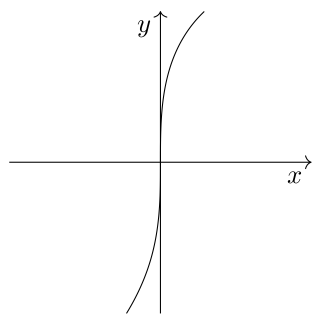
(b) At the origin, $y = 0$, so, since $(1 - 2x) \neq 0$, $\frac{dx}{dy} = 0$. The tangent is parallel to the y axis.

(c) Differentiating again by the product rule,

$$\begin{aligned} 3y^2 &= (1 - 2x)\tfrac{dx}{dy} \\ \Longrightarrow 6y &= -2\left(\tfrac{dx}{dy}\right)^2 + (1 - 2x)\tfrac{d^2x}{dy^2}. \end{aligned}$$

At the origin, $x = y = \frac{dx}{dy} = 0$, so the second derivative is zero. And, when moving along the curve through O, neither $-2(\frac{dx}{dy})^2$ nor $(1 - 2x)$ changes sign. But $6y$ does. Hence, the second derivative changes sign at O, signifying a point of inflection.

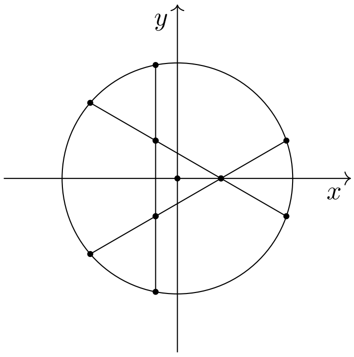
(d) Together with the above information, we note that, when $x > 0$, $y > 0$. So, the behaviour close to O is



3379. The LHS is a geometric series with first term 2^a, common ratio 2 and $b-a+1$ terms. Quoting the formula,

$$\begin{aligned} S &= \frac{2^a(2^{b-a+1}-1)}{2-1} \\ &\equiv 2^a \cdot 2^{b-a+1} - 2^a \\ &\equiv 2^{b+1} - 2^a, \text{ as required.} \end{aligned}$$

3380. Let the vertical chord, as shown in the question, be in the y direction, and put the origin at the centre.



The central equilateral triangle has side length 1, so its width in x is $\frac{\sqrt{3}}{2}$. Its centre divides its width in the ratio 1 : 2. Hence, the coordinates of the right-most points on the circumference are

$$\left(\tfrac{2}{3} \cdot \tfrac{\sqrt{3}}{2} + \tfrac{\sqrt{3}}{2}, \pm\tfrac{1}{2}\right).$$

Using Pythagoras, the squared distance r^2 from the origin to each of these is $\frac{7}{3}$. Hence, the area of the circle is $\frac{7}{3}\pi$.

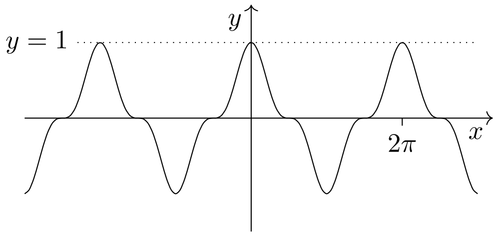
3381. Assume, for a contradiction, that $y = f(x)$ is both concave and increasing on $(0, \infty)$, and stationary at the origin.

Consider gradients at $x = 0$ and e.g. $x = 1$.

- We are told that $f'(0) = 0$. So, because f is an increasing polynomial for $x \in (0, \infty)$, we know that $f'(1) > 0$.
- We are also told that the second derivative is negative on $(0, 1]$, which, since $f'(0) = 0$, gives $f'(1) < 0$.

This is a contradiction. Therefore, no polynomial graph $y = f(x)$ can be both concave and increasing on $(0, \infty)$ and stationary at the origin. □

3382. Since the degree is odd, the graph looks broadly akin to that of $y = \cos x$: its maxima, minima and roots have the same coordinates. The difference is that single roots in the original graph have been converted to triple roots in the new graph. So, the points at which $y = \cos x$ crosses the x axis are now points of inflection:



3383. In each case, we (pre)multiply the probability of one specific successful outcome by the number of different versions of that successful outcome.

(a) $\mathbb{P}(\text{two pairs}) = {}^4\text{C}_2 \cdot 1 \cdot \frac{1}{6} \cdot \frac{5}{6} \cdot \frac{1}{6} = \frac{5}{36}$.

(b) $\mathbb{P}(\text{three of a kind}) = {}^4\text{C}_1 \cdot 1 \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{5}{6} = \frac{5}{54}$.

So, two pairs is more probable.

3384. Fixed points of $x_{n+1} = g(x_n)$ are roots of $x = g(x)$:

$$\begin{aligned} x &= \frac{1}{x-1} + \frac{1}{(x-1)^2} \\ \implies & x(x-1)^2 = (x-1) + 1 \\ \implies & x^2(x-2) = 0 \\ \implies & x = 0, 2. \end{aligned}$$

3385. ① If $a > 0$, then the equation is a quadratic in $\sqrt{x}$, whose roots (if they exist) are positive. So any value for $\sqrt{x}$ produces a value for x. Hence, we need $\Delta = 0$. This is $64 - 16a = 0$, so $a = 4$.

② If $a = 0$, then the equation isn't a quadratic. It is $-8\sqrt{x} + 4 = 0$, which has one root $x = \frac{1}{4}$.

③ If $a < 0$, then $\sqrt{64 - 16a} > 8$, so this produces one positive and one negative value for $\sqrt{x}$, thus exactly one value for x.

Hence, $a \in (-\infty, 0] \cup \{4\}$.

3386. (a) The string is inextensible.

(b) Assume that the two stacked blocks accelerate as one and that the frictional force between them is limiting. The reaction force between them is mg, so the total horizontal force on the upper block is μmg. So, $a = \mu g$. NII along the string, for the whole system, is

$$\begin{aligned} & Mg = (2m + M)\mu g \\ \implies & Mg - M\mu g = 2\mu mg \\ \implies & M(1-\mu) = 2\mu m \\ \implies & M = \frac{2\mu m}{1-\mu}. \end{aligned}$$

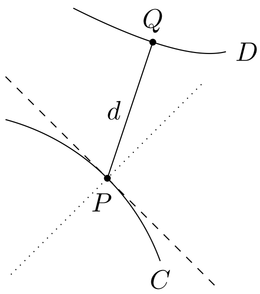
For M larger than this value, the lower block accelerates out from under the upper block. (Consider the tablecloth trick...)

(c) If $\mu \geq 1$, then, in the limiting case above, the acceleration of the connected blocks is

$$a = \mu g \geq g.$$

This is not possible: freefall acceleration is a non-attainable upper limit for such systems, whatever the masses involved. Hence, if $\mu \geq 1$, the limiting case above does not occur: the two stacked blocks *must* accelerate as one.

3387. Assume, for a contradiction, that the shortest path PQ is not normal to curve C at point P. Then the scenario is as follows, with the tangent and normal to curve C at P drawn.



At point P, curve C is approximated (perfectly in the infinitesimal limit) by the dashed tangent. Q lies off to the side of the dotted normal. So, as we move P along C in the direction of Q, d must decrease. But PQ is the shortest path between the curves. This is a contradiction. Hence, the shortest path is perpendicular to both curves. $\square$

3388. (a) Differentiating by the quotient rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{e^x x^3 + (e^x - e)3x^2}{x^6} \\ &= \frac{(x+3)e^x - 3e}{x^4}.\end{aligned}$$

(b) Substituting this into the LHS,

$$\begin{aligned}&\frac{dy}{dx} + \frac{3y}{x} \\ &= \frac{(x+3)e^x - 3e}{x^4} - \frac{3(e^x - e)}{x^4} \\ &\equiv \frac{xe^x}{x^4} \equiv \frac{e^x}{x^3}.\end{aligned}$$

So, the curve does satisfy the DE.

3389. Equating the differences,

$$\begin{aligned}&k^4 - k^3 = k^3 - k \\ \Longrightarrow\ &k^4 - 2k^3 + k = 0.\end{aligned}$$

Using a polynomial solver, $k = 0, 1, \frac{1}{2}\big(1 \pm \sqrt{5}\big)$.

3390. Using the chain rule,

$$\frac{dy}{dx} = -2xe^{-x^2}.$$

So, at $\left(p, e^{-p^2}\right)$, the tangent is

$$\begin{aligned}&y - e^{-p^2} = -2pe^{-p^2}(x - p) \\ \Longrightarrow\ &y = -2pe^{-p^2}x + \big(1 + 2p^2\big)e^{-p^2}.\end{aligned}$$

The y intercept is $\big(1 + 2p^2\big)e^{-p^2}$. Both quadratic and exponential factors are strictly positive for all x. Hence, no tangent passes through O. $\square$

3391. With a *smooth* pulley, the tension must be the same throughout the rope, so the two monkeys must rise at exactly the same rate, regardless of which one is doing the climbing.

With a *rough* pulley, the tension is greater on the side of the monkey doing the climbing, who duly ascends faster. In the limit of extreme roughness, the climbing monkey rises while the other monkey stays still.

3392. Multiplying out, we get two terms, each of which is a standard derivative:

$$\begin{aligned}&\int \sec x(\tan x + \sec x)\,dx \\ \equiv &\int \sec x \tan x + \sec^2 x)\,dx \\ = &\ \sec x + \tan x + c.\end{aligned}$$

3393. Using the definition of ${}^n\mathrm{C}_r$,

$$\begin{aligned}&{}^n\mathrm{C}_3 - 2 \cdot {}^n\mathrm{C}_2 = 12 \\ \Longrightarrow\ &\frac{n!}{3!(n-3)!} - 2\frac{n!}{2!(n-2)!} = 12 \\ \Longrightarrow\ &\frac{n(n-1)(n-2)}{6} - n(n-1) - 12 = 0 \\ \Longrightarrow\ &n^3 - 9n^2 + 8n - 72 = 0 \\ \Longrightarrow\ &n = 9.\end{aligned}$$

3394. (a) Setting $y = 0$ gives $\cos 2x = -1$. So, in $[0, 2\pi)$, $2x = \pi, 3\pi$. Hence, $x = \frac{\pi}{2}, \frac{3\pi}{2}$.

(b) The range of the numerator is $[0, 2]$ and the range of the denominator is $[1, 3]$. Hence, the quotient is non-negative.

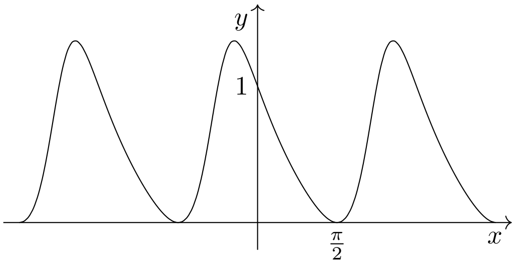
(c) By the quotient rule, the derivative is

$$\frac{-2\sin 2x(2 + \sin 2x) - (1 + \cos 2x)2\cos 2x}{(2 + \sin 2x)^2}.$$

The points midway between x intercepts are $x = 0, \pi, 2\pi$ etc. At $x = 0$ (or any other),

$$\frac{dy}{dx} = \frac{-4}{4} = -1.$$

(d) From (a) and (b), we know that the curve is tangent to the x axis. It is also periodic, with period π. And, halfway between x intercepts, the curve has gradient -1, which shows that each local maximum is closer to the x value below it than that above it:



3395. Call the people $A_1, A_2, B_1, B_2, \ldots$. We can place A_1 without loss of generality. The probability that A_2 sits opposite is $\frac{1}{7}$. Then we can place B_1 wlog. The probability that B_2 sits opposite is $\frac{1}{5}$. The pattern continues, giving

$$p = \tfrac{1}{7} \cdot \tfrac{1}{5} \cdot \tfrac{1}{3} \cdot 1 = \tfrac{1}{105}.$$

——— Alternative Method ———

There are 8! ways in which the people can sit down. For successful outcomes, there are 4! locations in which each couple can sit, combined with 2^4 ways in which the partners can sit. This gives

$$p = \frac{4! \times 2^4}{8!} = \tfrac{1}{105}.$$

3396. (a) Vertically, $-2 = (5\sin\theta)t - 5t^2$. Horizontally, $2 = (5\cos\theta)t$. Substituting for t,

$$\begin{aligned} -2 &= 5\sin\theta \cdot \frac{2}{5\cos\theta} - 5 \cdot \frac{4}{25\cos^2\theta} \\ \implies & 2\sec^2\theta - 5\tan\theta - 5 = 0. \end{aligned}$$

(b) Using the second Pythagorean trig identity,

$$\begin{aligned} & 2(1+\tan^2\theta) - 5\tan\theta - 5 = 0 \\ \implies & 2\tan^2\theta - 5\tan\theta - 3 = 0 \\ \implies & (2\tan\theta + 1)(\tan\theta - 3) = 0. \end{aligned}$$

We need the positive angle, so $\theta = \arctan 3$.

3397. Let k be the mth digit of the decimal expansion of x. Then x can be expressed as $x = X + k \times 10^{-m}$, where k is non-zero and the decimal expansion of X terminates before the mth digit. This gives

$$x^n = \left(X + k \times 10^{-m}\right)^n.$$

Expanding this binomially, the last term, which contains the last decimal digit, is $k^n \times 10^{-mn}$. Since k^n cannot have last digit zero, the last digit of this number is in the mnth place. □

3398. Let the tiles have area 3. The resulting square must have area $4 \times 3 = 12$: its side length must be $\sqrt{12}$. But $\sqrt{12} \notin \mathbb{Z}$, so cannot be constructed from the unit side lengths of the tiles. QED.

3399. The first graph is a circle, radius 3, centred at the origin: the Cartesian equation is $x^2 + y^2 = 9$. The second graph is a straight line, through $(5,1)$, with gradient $\frac{1}{2}$: the Cartesian equation is $y = \frac{1}{2}x - \frac{3}{2}$. Solving these simultaneously gives $(3,0)$, as in the question, and $(-1.8, -2.4)$.

3400. By the product and chain rules:

$$\begin{aligned} & \frac{d}{dx}\Big(\tfrac{1}{2}x\big(\sin(\ln x) + \cos(\ln x)\big)\Big) \\ \equiv\ & \tfrac{1}{2}\big(\sin(\ln x) + \cos(\ln x)\big) \\ & \quad + \tfrac{1}{2}x\big(\cos(\ln x) \cdot \tfrac{1}{x} - \sin(\ln x) \cdot \tfrac{1}{x}\big) \\ \equiv\ & \tfrac{1}{2}\big(\sin(\ln x) + \cos(\ln x)\big) \\ & \quad + \tfrac{1}{2}\big(\cos(\ln x) - \sin(\ln x)\big) \\ \equiv\ & \cos(\ln x). \end{aligned}$$

This verifies the given integral.

——— End of 34th Hundred ———